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# HYDRODYNAMIC CHARACTERISTICS OF A WING IN A STRATIFIED FLUID NEAR THE BOTTOM

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*The problem of the motion of a thin wing in a stratified fluid near the bottom is considered. A solution is found using the logarithmic dynamic potential. The dependence of the hydrodynamic force and moment on the input parameters, namely, the Strouhal and Froude numbers and the distance to the bottom, is studied. An important feature of the amplitudes of nonstationary loads on the wing is their nonmonotonic character in the case where the frequency of vibrations is lower than the Brunt–Väisälä frequency, which is explained by the interaction between the wing's vibrations and the internal waves reflected from the bottom.*

The effect of the bottom on the hydrodynamic characteristics of a wing in a uniform fluid has been investigated in many studies. Basin and Shadrin [1] summarized the results obtained. The author considered the unsteady motion of a wing and the force acting on it in a uniformly stratified unbounded fluid [2, 3]. In the present paper, the unsteady hydrodynamic characteristics of a wing in a stratified fluid near the bottom are analyzed.

**1.** As in [2], we shall model a wing by an infinitely thin plate of length $2c$ ($c$ is the half-chord of the wing) located a distance $h$ from the bottom. We introduce the Cartesian coordinate system $(x, y)$, directing the $x$ axis along the bottom. The wing moves horizontally at a constant velocity $V$. At the moment $t = 0$, the projection of the wing onto the $Ox$ axis occupies the segment $[-c, c]$ (Fig. 1). At this moment the wing begins to undergo small vibrations in the transverse direction in accordance with a specified law:

$$v_2(x,t) = f(x,t), \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity.

The stratification is assumed to be weak and exponential in character:

$$\rho_0(y) = \rho_* \exp(-\beta y), \qquad \beta = \text{const}.$$

It is also assumed that the dimensions of the wing are small compared with the characteristic dimension of the stratification, i.e., $c\beta \ll 1$. According to these assumptions, it is possible to use the Boussinesq approximation. As a result, for the stream function $\psi(\mathbf{x}, t)$ $[\mathbf{x} = (x, y)]$ we obtain the Sobolev equation

$$\frac{\partial^2 \Delta\psi}{\partial t^2} + \omega_0^2 \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{1.2}$$

where $\omega_0^2 = -g\rho_0'/\rho_0$ is the Brunt–Väisälä frequency.

Thus, the liquid motion satisfies Eq. (1.2) outside the wing and the wake for $t > 0$, the initial conditions

$$\psi(\mathbf{x}, 0) = \psi_t(\mathbf{x}, 0) = 0, \tag{1.3}$$

and the following boundary conditions: the condition of nonflow on the wing (1.1) and at the bottom

$$v_2(x, 0, t) = 0,$$

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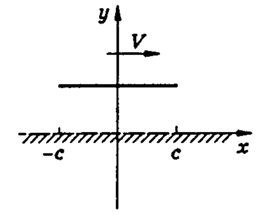


Fig. 1. Wing near the bottom.

the conditions in the wake

$$[v_2] = [p] = 0, \tag{1.4}$$

and the Kutta–Joukowski conditions at the trailing edge. Here the square brackets denote the difference in the corresponding quantities above and below the contour. In addition, the function $\psi(\mathbf{x}, t)$ should be subject to the regularity conditions at infinity $|\nabla\psi| = O(r^{-2})$, $r = \sqrt{x^2 + y^2}$ $(r \to \infty)$ and at the leading edge $|\nabla\psi| = O(r_1^{-\delta})$ $(0 < \delta < 1)$, where $r_1$ is the distance to the leading edge.

2. We use the reflection method and extend the function $\psi(\mathbf{x}, t)$ to the lower half-plane in an odd manner. We seek the stream function in the form of a logarithmic dynamic potential [2–4]. Changing over to the coordinate system $x', y'$ attached to the wing

$$x = x' + Vt, \qquad y = y',$$

we obtain for the stream function $\Psi'(\mathbf{x}', t) = \psi(\mathbf{x} + Vt\mathbf{e}_1, t)$ the representation (henceforth, the primes are omitted)

$$\Psi(\mathbf{x}, t) = \int_{l_t} \nu(\xi, t) \ln|\mathbf{x} - \mathbf{y}(\xi)| d\xi + \int_0^t d\sigma \int_{l_\sigma} \nu(\xi, \sigma) \frac{1}{t - \sigma} \left[1 - \cos\left(\omega_0 (t - \sigma) \frac{y - h}{|\mathbf{x} + V(t - \sigma)\mathbf{e}_1 - \mathbf{y}(\xi)|}\right)\right] d\xi$$

$$- \int_{l'_t} \nu(\xi, t) \ln|\mathbf{x} - \mathbf{y}(\xi)| d\xi - \int_0^t d\sigma \int_{l'_\sigma} \nu(\xi, \sigma) \frac{1}{t - \sigma} \left[1 - \cos\left(\omega_0 (t - \sigma) \frac{y + h}{|\mathbf{x} + V(t - \sigma)\mathbf{e}_1 - \mathbf{y}(\xi)|}\right)\right] d\xi. \tag{2.1}$$

Here $l_t = l_0 + l_{1t}$ is the contour of the original wing $l_0 = (-c, c)$ and the wake $l_{1t} = (-c - Vt, -c)$ at the moment of time $t$, $l'_t$ is the reflected contour, $\nu(\xi, t)$ is the density of the potential, and $\mathbf{e}_1$ is the unit vector along the $x$ axis. The following relations, which relate the density of the potential in the wake and the density on the wing:

$$\nu(\xi, t) = \gamma\left(t + \frac{\xi + c}{V}\right) J_0\left(\omega_0 \frac{\xi + c}{V}\right) + h(\xi, t), \xi \in l_{1t} \tag{2.2}$$

were derived in [2] from the Kutta–Joukowski conditions and (1.3). Here the function $\gamma(t)$ has the meaning of the velocity jump at the trailing edge, as in a uniform fluid:

$$\gamma(t) = [v_1(-c, t)] = -\frac{1}{V} \frac{d\Gamma_0}{dt}; \tag{2.3}$$

$\Gamma_0(t)$ is the circulation of the velocity around the wing:

$$\Gamma_0(t) = \int_{-c}^{c} \nu(\eta, t) d\eta + \omega_0 \int_{-c}^{c} d\eta \int_{\eta}^{c} \nu\left(\xi, t + \frac{\eta - \xi}{V}\right) \theta\left(t + \frac{\eta - \xi}{V}\right) G\left(\omega_0 \frac{\xi - \eta}{V}\right) d\xi; \tag{2.4}$$

$h(\xi, t)$ is an additional term owing to the stratification:

$$h(\xi, t) = -\frac{\omega_0}{V} \int_{-c}^{c} \nu\left(\eta, t + \frac{\xi - \eta}{V}\right) \left[J_1\left(\omega_0 \frac{\eta - \xi}{V}\right) + \omega_0 \int_0^{(\eta + c)/V} J_1\left(\omega_0 \frac{\eta - \xi}{V} - \tau\right) G(\omega_0 \tau) \, d\tau\right] d\eta, \tag{2.5}$$

$$G(x) = \int_0^x \frac{J_1(y)}{y}\,dy$$

($J_0$ and $J_1$ are Bessel functions and $\theta$ is the Heaviside function). From (2.1) and the nonflow condition on the wing we have

$$f(x,t) = \int_{l_t} \nu(\xi,t)\Big[\frac{1}{\xi - x} - \frac{\xi - x}{(x-\xi)^2 + 4h^2}\Big]\,d\xi + 2h\omega_0 \int_0^t d\sigma \int_{l_\sigma} \nu(\xi,\sigma)$$

$$\times \sin\Big(\omega_0(t-\sigma)\frac{2h}{\sqrt{(x-\xi+V(t-\sigma))^2+4h^2}}\Big)\frac{(x-\xi+V(t-\sigma))\,d\xi}{[(x-\xi+V(t-\sigma))^2+4h^2]^{3/2}}. \tag{2.6}$$

Thus, for the unknown density of the potential $\nu(\xi,t)$ we obtain the singular integral equation (2.6), relations (2.3)–(2.5) in the wake, and the initial conditions $\nu(\xi,0) = \nu_t(\xi,0) = 0$ for $\xi \in l_0$.

**3.** We consider the question of the asymptotics of the solution for $t \to \infty$ in the case of harmonic vibrations of a wing with frequency $\omega$, $f(x,t) = f(x)e^{i\omega t}$. For this purpose, we apply the Laplace transform in time:

$$\nu^L(\xi,p) = \int_0^\infty e^{-pt}\nu(\xi,t)\,dt.$$

From (2.6) we then have

$$\int_{-\infty}^{c} \nu^L(\eta,p)\Big[\frac{1}{\eta - x} - \frac{\eta - x}{(\eta - x)^2 + 4h^2}\Big]\,d\eta + \omega_0 2h \int_{-\infty}^{c} \nu^L(\xi,p)\,d\xi \int_0^\infty e^{-p\sigma}$$

$$\times \sin\Big(\omega_0\sigma\frac{2h}{\sqrt{(x+V\sigma-\xi)^2+4h^2}}\Big)\frac{(x+V\sigma-\xi)}{[(x+V\sigma-\xi)^2+4h^2]^{3/2}}\,d\sigma = f^L(\eta,p). \tag{3.1}$$

It was shown in [2] that, for harmonic vibrations of a wing in an unbounded weakly stratified liquid ($s_0 \ll 1$), steady vibratory motion of the liquid is observed in a coordinate system attached to the wing at $t \to \infty$. A similar fact is true in the presence of a bottom. What follows is a brief proof. We shall change over from the singular equation (3.1) and relations (2.2)–(2.5) to a Fredholm integral equation.

The kernel of the equation is of complicated form and it is not displayed here. In the case of an unbounded fluid, the corresponding kernel is presented in [2]. In the presence of a bottom, this equation is supplemented by terms that correspond to the reflected wing. The kernel depends on two dimensionless parameters: the complex parameter $z = pc/V$, which is spectral, and the real parameter $s_0 = \omega_0 c/V$, which is the inverse Froude number for internal waves. The kernel is an analytic function of $z$ in the entire complex plane, except for the segment on the imaginary axis $[-is_0, is_0]$.

Let $h$ be arbitrary but fixed. It follows from the general theory [5] that the Fredholm equation is uniquely solvable throughout the complex plane $z$, except, possibly, for a countable number of poles of the resolvent. We shall show that the resolvent has no poles in the right-hand half-plane. We consider the solution of this equation and perform an inverse Laplace transformation. Since perturbed flow does not occur at $t < 0$. there are no poles in the right-hand half-plane of the resolvent. We shall show that the resolvent has no poles on the imaginary axis as well. For $z = O(s_0)$, the kernel in the leading order as $S_0 \to 0$ (the principal part of the kernel) corresponds to steady motion of a wing in a uniform fluid near the bottom [1]. After the principal part of the kernel is inverted, we obtain a Fredholm equation with the operator $I + T_1(z, s_0)$, where $\|T_1\| = O(s_0)$. Hence, $T_1$ is a compressing operator, and the solution is unique. For $z \gg s_0$, the principal part of the kernel corresponds to unsteady motion of a wing in a uniform fluid near the bottom. Inverting the principal part of the kernel, we again obtain a compressing operator.

Thus, for small $s_0$ and Re $z \geqslant 0$, the problem has a unique solution, and the poles of the resolvent can lie only in the left-hand half-plane. As $s_0$ grows, the poles (if they were in the left half-plane) can.

generally speaking, approach the imaginary axis. New poles can appear only on the boundary of the domain of analyticity of the kernel, i.e., on the segment $[-is_0, is_0]$. Since the right-hand part has a pole for $p = i\omega$, the solution also has a pole at this point. Next, writing the Bromwich integral, in accordance with the residue theorem we obtain a harmonic solution for $t \to \infty$. The statement is proved.

We introduce dimensionless coordinates $\xi$ and $\eta$ referred to the half-chord of the wing $c$:

$$\xi = x/c, \qquad \eta = y/c.$$

We define the functions $æ_*(\xi) = \nu^L(c\xi, i\omega)$, $f_*(\xi) = f^L(c\xi, i\omega)$ and the dimensionless parameters $h' = h/c$ and $s = \omega c/V$, i.e. the Strouhal number (the primes will be omitted). We then obtain the singular integral equation

$$\int_{-\infty}^{1} æ_*(\zeta)\Big[\frac{1}{\zeta-\xi} - \frac{\zeta-\xi}{(\zeta-\xi)^2+4h^2}\Big]\,d\zeta + 2hs_0 \int_{-\infty}^{1} æ_*(\zeta)\,d\zeta$$

$$\times \int_0^{\infty} e^{-is\tau} \sin\Big(s_0\tau \frac{2h}{\sqrt{(\xi-\zeta+\tau)^2+4h^2}}\Big) \frac{(\xi-\zeta+\tau)}{[(\xi-\zeta+\tau)^2+4h^2]^{3/2}}\,d\tau = f_*(\xi). \tag{3.2}$$

Here the density in the wake is as follows:

$$æ_*(\xi) = -is_0 J_0(s_0(\xi+1))e^{is(\xi+1)} \int_{-1}^{1} æ_*(\zeta)\Big[1 + s_0 \int_0^{1+\zeta} e^{-is\sigma} G(s_0\sigma)\,d\sigma\Big]\,d\zeta$$

$$- s_0^2 \int_{-1}^{1} æ_*(\zeta)e^{-is(\zeta-\xi)}\Big[J_1(s_0(\zeta-\xi)) + s_0 \int_0^{1+\zeta} J_1(s_0(\zeta-\xi-\sigma))G(s_0\sigma)\,d\sigma\Big]\,d\zeta. \tag{3.3}$$

**4.** Equation (3.2) together with relation (3.3) was solved numerically by the discrete-vortex method [6] using the Belotserkovskii scheme: a vortex in the 1/4 segment and a control point in the 3/4 segment from the leading edge.

After we find the density of the potential $æ_*(\xi)$, the amplitudes of the nonstationary forces and the moment are determined by the following formulas [2]:

$$Y_* = -2\pi\rho_0 Vc\Big[(1+is)\int_{-1}^{1} æ_*(\xi)\,d\xi + is\int_{-1}^{1} \xi æ_*(\xi)\,d\xi + s_0 \int_{-1}^{1} æ_*(\xi)\,d\xi$$

$$\times \int_0^{\xi+1} e^{-is\zeta} G(s_0\zeta)\,d\zeta + iss_0 \int_{-1}^{1} æ_*(\xi)\,d\xi \int_0^{1+\xi} (1+\xi-\zeta)e^{-is\zeta}G(s_0\zeta)\,d\zeta\Big],$$

$$M_* = -2\pi\rho_0 Vc^2\Big[\int_{-1}^{1} \xi æ_*(\xi)\,d\xi - \frac{is}{2}\int_{-1}^{1} æ_*(\xi)(1-\xi^2)\,d\xi + s_0 \int_{-1}^{1} æ_*(\xi)\,d\xi$$

$$\times \int_0^{\xi+1} (\xi-\zeta)e^{-is\zeta}G(s_0\zeta)\,d\zeta + iss_0 \int_{-1}^{1} \xi d\xi \int_{\xi}^{1} d\zeta \int_{\zeta}^{1} æ_*(\eta)e^{is(\zeta-\eta)}G(s_0(\eta-\zeta))\,d\eta\Big].$$

The integral

$$\int_0^{\infty} e^{-is\tau} \sin\Big(s_0\tau \frac{2h}{\sqrt{(\xi+\tau-\zeta)+4h^2}}\Big) \frac{(\xi+\tau-\zeta)}{[(\xi+\tau-\zeta)^2+4h^2]^{3/2}}\,d\tau \tag{4.1}$$

for $s \neq 0$ was divided into two segments $[0, \tau_n]$ and $[\tau_n, \infty]$. This integral was calculated on the segment $[0, \tau_n]$ according to the scheme of rectangles with a point at the middle of it and by integration by parts on

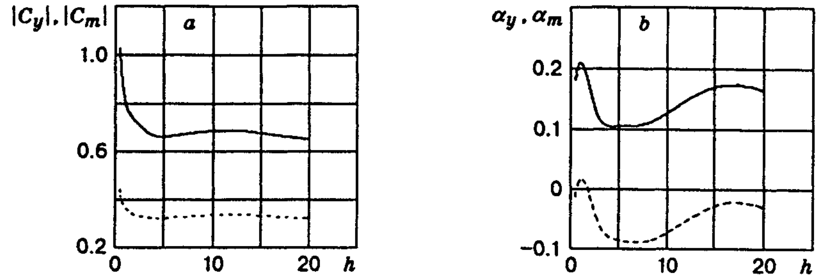


Fig. 2. Dependence of the modulus (a) and phase (b) of the unsteady force and moment on the distance to the bottom $h$ for $s = 0.2$ and $s_0 = 0.5$.

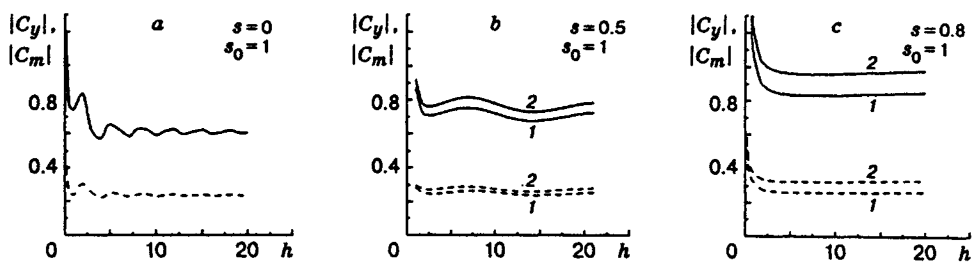


Fig. 3. Dependence of the force and moment moduli on the distance to the bottom $h$ for $s_0 = 1$ and various values of $s$.

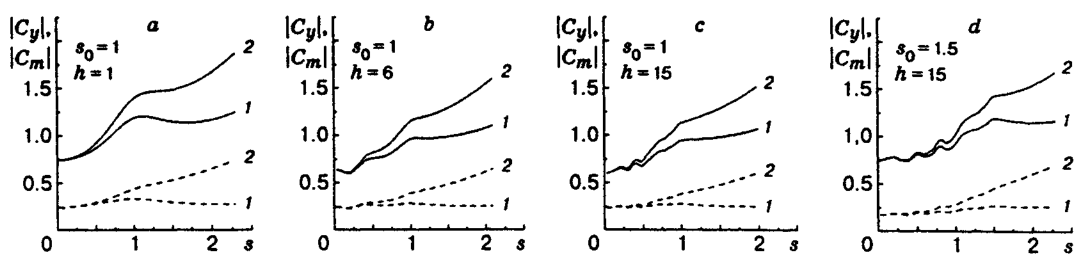


Fig. 4. Dependence of the force and moment moduli on the Strouhal number $s$ for various distances of the wing from the bottom and various Froude numbers.

the segment $[\tau_n, \infty]$. For $s = 0$, we made the following substitution: $\sigma = 2h\tau/\sqrt{(\xi + \tau - \zeta)^2 + 4h^2}$. Here the domain of integration becomes finite ($0 < \sigma < \sigma_{\max}$):

$$\sigma_{\max} = \begin{cases} 2h & \text{for} \quad (\xi - \zeta) \geqslant 0, \\ \sqrt{4h^2 + (\xi - \zeta)^2} & \text{for} \quad (\xi - \zeta) < 0. \end{cases}$$

Integral (4.1) determines the interaction of the vibrations of the wing and the reflected internal waves. Numerical computations showed that, for $s < s_0$, the integral behaves nonmonotonically relative to $h$. The unsteady forces and the moments acting on the wing behave correspondingly.

Computations were performed for two cases, namely, bending and torsional vibrations of the wing. Here the displacement of the wing was specified in the form

$$w_1(x,t) = y_0 e^{i\omega t}, \qquad w_2(x,t) = \alpha_0 x e^{i\omega t},$$

where $y_0$ and $\alpha_0$ are the dimensionless amplitudes of vibrations of the wing. We introduce the dimensionless force $C_y$ and moment $C_m$ coefficients, which are related to $Y_{i*}$ and $M_{i*}$ by the relations

$$Y_{i*} = 2\pi\rho_0 V^2 c a_i C_y, \quad M_{i*} = 2\pi\rho_0 V^2 c^2 a_i C_m, \quad i = 1, 2.$$

Here the subscript $i$ indicates the type of vibration of the wing, and $a_i$ are the normalization coefficients: $a_1 = isy_0$ and $a_2 = \alpha_0$.

Figure 2 shows the moduli and phases of the dimensionless unsteady forces $C_y$ and moments $C_m$ for bending vibrations of the wing for $s = 0.2$ and $s_0 = 0.5$. In this figure and the subsequent ones, the solid curves refer to $C_y$ and $\alpha_y$ and the dashed curves refer to $C_m$ and $\alpha_m$. The corresponding curves for torsional vibrations of the wing differ little and, therefore, we omit them.

Figure 3 shows force and moment moduli for $s_0 = 1$ [(a) refers to steady motion of the wing at a small angle of attack $\alpha_0$, i.e., $s = 0$, and (b) and (c) refer to $s = 0.5$ and 0.8 for bending and torsional vibrations of the wing (curves 1 and 2)]. It follows from the results obtained that the oscillations of the amplitude of harmonic loads relative to $h$ depend on the relation between $s$ and $s_0$. When $s$ approaches $s_0$, the curves becomes increasingly flat, and they behave monotonically for $s > s_0$, because there are no internal waves, and the solution becomes an asymptote very rapidly.

The force and moment moduli versus the Strouhal number are shown in Fig. 4 for various values of $h$. For $s = s_0$, the flow regimes change: internal waves occur for $s < s_0$ and these waves are absent for $s > s_0$. The curves are not smooth for $s = s_0$. It is seen from Fig. 4 that, for $s > s_0$, the solution behaves in the same manner for various values of $h$, since it acquires the asymptotic character very rapidly. Meanwhile, for $s < s_0$, the solution becomes oscillating increasingly rapidly with increasing $h$. A similarity to acoustics is observed: the larger the width of the channel, the shorter the wavelength and the larger the number of crests and troughs. As in acoustics, it is possible that eigenvalues of the homogeneous problem are present in the left half-plane for $s < s_0$.

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